Technical Comments

Comment on "Long-Term Evolution of Near-Geostationary Orbits"

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REFERENCE 1 describes the development of a single-averaging theory and its application to the prediction of a near-synchronous orbit. Single averaging usually denotes a formulation in which the high-frequency perturbations associated with the satellite period and the Earth's rotational period are removed from the equations of motion. Thus, step sizes on the order of one day are allowed in the integration of the averaged equations of motion.

The purpose of this Comment is to connect the results in Ref. 1 with results that have already appeared in the literature. In many cases, the author of Ref. 1 gives a limited result when a more general result is available; these instances are noted. We also discuss the construction of initial conditions for a single-averaged orbit prediction. Finally, some aspects of the very long-term motion of desynchronized orbits (such as GEOS-2) are discussed.

Fundamental to the development of this single-averaged theory are the differential equations for the motion of the equinoctial elements due to a disturbing potential. These are Eqs. (3) in Ref. 1. These same equations were given much earlier in Refs. 2 and 3.

The disturbing potential due to third bodies (moon and sun) is developed in Ref. 1 with the aid of a Poisson series symbolic algebra program.[4] The potentials include "up to second order eccentricity terms."[1] The author of Ref. 1 notes that "lengthy computer-generated secular terms can be rearranged in extremely compact forms" by introducing the C and S auxiliary parameters [Eq. (17) in Ref. 1]. The averaged potentials are obtained under the assumption that the third body positions are held constant during the averaging operation. Similar potentials were given much earlier in Refs. 3 and 5. In particular, Ref. 5 gave the following general form for the third body potential in terms of equinoctial elements:

$$\bar{U}^{(3)} = \frac{\mu_3}{R_3} \sum_{m=0}^{M} (-1)^m \kappa_m G_m \sum_{n}^{N} F_n^m \tag{1}$$

$$n = \begin{cases} 2 \text{ for } m = 0,1 \\ m \text{ for } m \geq 2 \end{cases}$$

where

$$\kappa_m = \begin{cases} 1 \text{ for } m = 0 \\ 2 \text{ for } m \geq 1 \end{cases}$$

In Eq. (1) the quantity G_m is a simple function of the equinoctial elements f and g and the direction cosines C and S (Ref. 1 notation). The G_m are proportional to the mth power of the eccentricity. The m summation upper limit is less than or equal to N, the maximum value of the parallax index. The F_n^m are given by

$$F_n^m = \left(\frac{a}{R_3}\right)^n V_{n,m} Q_{n,m}(\gamma) A_{n+2}^m \tag{2}$$

In Eq. (2), the quantities V, Q, and A all are governed by very simple recursions. The quantity γ is the dot product of the unit vector to the third body with the unit normal to the orbital plane. The function A_{n+2}^m is the kernel of the Hansen coefficient with subscript zero and is closed form with respect to eccentricity. Equations (1) and (2) form the basis for an extremely compact (230 source instructions) recursive Fortran implementation of the single-averaged equations of motion due to the third-body perturbations.[6] Because the outer sum in Eq. (1) is proportional to the eccentricity, the Ref. 6 implementation can be truncated *at execution time* for small eccentricity. However, the same code can also be used for high-eccentricity orbits (such as those listed in Ref. 7), where the parallax factor a/R_3 offers the main truncation. From an overall point of view, the recursive code offers significant advantage in flexibility and efficiency; the Poisson series analysis described in Ref. 1 would have to be revisited each time the orbital type is changed. The Poisson series analytical results, however, are useful for providing independent tests for the recursive algorithms.

Reference 1 next gives explicit analytical formulations for the disturbing potentials due to the zonal harmonics J_2, J_3, and J_4, in terms of the equinoctial variables. These potentials also include up to second-order eccentricity terms. Similar to the situation with the third body, Ref. 5 much earlier gave a general, recursive formulation for the averaged potential due to zonal harmonics in equinoctial elements [Eq. (118)]. This formulation is closed-form in the eccentricity; the recursive flow again provides for systematic truncation on the eccentricity at execution time. The above comments about the flexibility of the recursive approach still apply.

Reference 1 also formulates a tesseral resonance potential for the synchronous orbit in the equinoctial elements truncated for small eccentricity. In contrast, Ref. 8 [*Eq. (43)*] gives a general formulation of the resonance potential in terms of special functions of the equinoctial elements. These special functions are governed by recursion relations for Jacobi polynomials[9] and Hansen coefficients.[10] Recursive evaluations of the Ref. 8 resonance potential were devised by Dunham (see Ref. 6 for discussion) and improved subsequently by Proulx.[11] This work of Proulx included an application of a long-lost recursion for the Hansen coefficients.[10] Proulx and McClain[12] also developed a modified Hansen coefficient expansion with improved eccentricity convergence. Particularly with tesseral resonance where there are multiple commensurability constraints in common applications, it seems desirable to avoid the modification of the theory for each orbit type implied by the approach of Ref. 1.

Reference 1 establishes initial conditions and reference values for comparison by "averaging 48 half hourly values of the orbital elements produced by a different orbit generator containing all short-periodic contributions." This procedure is problematical for the precise determination of mean elements because it implies restrictive assumptions about the short-periodic frequency context. Alternative procedures (Refs. 13 and 14) require a short-periodic model in the equinoctial elements compatible with the averaged equations of motion.

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Such a model in the equinoctial elements is given in Refs. 15–17. Green[15] considered the effect of holding the moon's position constant during one satellite orbit. He noted that the major impact is in the short-periodic motion. While Ref. 1 emphasizes the moon's motion as a source of second-order coupling, there is also the possibility of coupling between the J_2 secular effects and the shallow tesseral resonance oscillations that occur in near-geosynchronous orbits.

Finally, a double-averaging theory (see Collins[18]) is also an appropriate tool for analyzing the very long-term motion of near-geostationary orbits.[19]

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Reply by Author to P. Cefola

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THE author is grateful to Dr. Cefola for his keen interest in Ref. 1 and for pointing out the connection between the results of Ref. 1 and other formulations. It is unfortunate that essentially all references quoted by Dr. Cefola are either conference papers or internal notes, which limits their dissemination considerably. Perhaps it would be of interest to summarize their principal results in an archival publication.

When Ref. 1 is compared with other results, it should not be overlooked that the scope of Ref. 1 was limited to providing useful results for just one specific, but extremely important, class of orbits. No attempt at generalization of the results of Ref. 1 was made. It is believed that a practicing engineer dealing with geostationary orbits would prefer the explicit closed-form results of Ref. 1 to the more general but recursive formulation suggested by Dr. Cefola.

Concerning the use of the equinoctial elements, it is felt that proper credit was given to earlier work by referring to the archival publication by Broucke and Cefola (Ref. 12 in Ref. 1). The derivation of Eqs. (3) in Ref. 1 was, in fact, carried out independently using Campbell's formulation (details are described in an internal ESOC report[2]).

The form of Dr. Cefola's general result for the averaged third-body potential, as described in his Eqs. (1) and (2), shows complete agreement with Eqs. (18–20) of Ref. 1. Whereas the recursive code, as advocated by Dr. Cefola, would offer advantages from an overall generality and flexibility of point of view, the explicit results of Ref. 1 are of more practical value for the specific case of a near-geostationary orbit. The remark by Dr. Cefola that the Poisson series analysis must be revisited each time when the orbital-type changes should be seen in the same light.

The comments by Dr. Cefola on the zonal and tesseral harmonics formulation are of the same nature as those on the potential development and can therefore be answered by the same argument.

The procedure for obtaining initial conditions for the mean elements that was adopted in Ref. 1 is rather straightforward since it was needed only for establishing the accuracy of the long-term model and not for an accurate orbit prediction. The improvements in this procedure suggested by Dr. Cefola could have a slightly beneficial effect on the accuracies quoted in Table 3 of Ref. 1.

Finally, it is noted that the verification of the coupling between J_2 secular effects and shallow tesseral resonances would require a number of controlled simulation runs and cannot be commented on now.

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